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Goldstone Boson Normal Coordinates in Interacting Bose Gases

T. Michoel[1], A. Verbeure[2]

Instituut voor Theoretische Fysica
Katholieke Universiteit Leuven
Celestijnenlaan 200D
B-3001 Leuven, Belgium

Abstract

For the phenomenon of Bose-Einstein condensation we construct the canonical pair of field operators of the Goldstone Bosons explicitly as fluctuation operators in the ground state. We consider the imperfect Bose gas as well as the weakly interacting Bose gas. We prove that a canonical pair of fluctuation operators is always related to the order parameter and the generator of the broken symmetry fluctuations. We find that although the first one has an anomalous behaviour, the second one is squeezed by the same inverse rate. Furthermore, we prove that this canonical pair separates from the other variables of the system and that it behaves dynamically as oscillator variables. Finally the long wavelength behaviour of the spectrum determines the lifetime of this pair.

[1] Aspirant van het Fonds voor Wetenschappelijk Onderzoek - Vlaanderen
[1] Email: `tom.michoel@fys.kuleuven.ac.be`
[2] Email: `andre.verbeure@fys.kuleuven.ac.be`

1 Introduction

The phenomenon of spontaneous symmetry breaking (SSB) is a representative tool for the explanation of many phenomena in modern physics of field theory and statistical mechanics of many-body theory. The analysis of SSB goes back to the Goldstone Theorem [1], which has been the subject of much analysis. It is proved that for short range interactions in many-body systems SSB implies the absence of an energy gap in the excitation spectrum [2, 3].

For long range interactions the SSB has also been studied extensively. In the physics literature the phenomenon is known as the occurence of oscillations with frequency spectrum taking a finite value $\omega \neq 0$ at $k = 0$ [4, 5, 6]. Different approximation methods, typical here is the random phase approximation, yield the exact computation of these frequencies. For some mean field models, the BCS-model [7], the Overhauser model [8], the anharmonic crystal model [9], and for the Jellium model [10], we were able to give the mathematical status of these frequencies as elements of the spectrum of typical fluctuation operators (see [11, 12]).

The typical operators entering in the discusssion are the generator of the broken symmetry and the order parameter. In physical terms expressed, it is the charge or density operator and the current operator. Their fluctuation operators form a quantum canonical pair, which decouples from the other degrees of freedom of the system. As fluctuation operators are collective operators, they describe the collective mode accompanying the SSB phenomenon. Hence for long range interacting systems, we realised mathematically rigorously in these models, the so-called Anderson theorem [13, 14] of 'restauration of symmetry', stating that there exists a spectrum of collective modes $\omega(k \to 0) \neq 0$ and that the mode in the limit $k \to 0$ is the operator which connects the set of degenerate temperature states, i.e. 'rotates' one ergodic state into an other. We conjecture that our results of [7, 8, 10] can be proved for general long range two-body interacting systems as a universal theorem. Anderson did formulate his theorem in the context of the Goldstone theorem for short range interacting systems, i.e. in the case $\omega(k \to 0) = 0$ of absence of an energy gap in the ground state.

Of course one knows that there is no one-to-one relation between short range interactions and the absence of an energy gap for symmetry breaking systems (see e.g. [9]). The imperfect Bose gas is an example of a long range interacting system showing SSB, but without energy gap. In this paper we realise the above described program of construction of the collective modes operators of condensate density and condensate current, as normal modes dynamically independent from the other degrees of freedom of the system. We consider

the whole temperature range, the ground state included.

In particular the ground state situation is interesting, because it yields a non-trivial quantum mechanical canonical pair of conjugate operators, giving an explicit representation of the field variables of the socalled Goldstone boson. One can consider this result as a formal step forward beyond the known analysis of the Goldstone phenomenon.

Moreover in Section 4, we extend this result tot the weakly interacting Bose gas of superfluidity. It is interesting to remark that here the situation is intrinsically different in the sense that only the condenste density mode is spontaneously broken. One checks explicitly that the density fluctuation operator and the order parameter fluctuation operator do not form a non-trivial pair, but the condensate density and the order parameter fluctuation operators do.

Hence in both models the fluctuation operators of the generator of the broken symmetry and of the order parameter form a non-trivial canonical pair. The latter one shows off-diagonal long range order, therefore the density-density correlation can not share this property. This can be interpreted as that a spontaneously broken symmetry behaves like an approximate symmetry. The explicit construction of the canonical pair amounts to the realisation of 'restauration of symmetry', an idea put forward by Anderson [13, 14].

Furthermore, for both models, we prove that the canonical pair of Goldstone fluctuation modes separates dynamically from the other variables of the system and behaves like harmonic oscillator modes with a frequency proportional to the condensate density, i.e. this phenomenon disappears if no condensation is present. It turns out that this pair of variables has a lifetime in the long wavelength limit which is determined by the long wavelength behaviour of the spectrum of the system.

2 Fluctuation operators

We want to study different models of a Bose gas in which there is breaking of the gauge symmetry. In general, a system of identical bosons of mass m in a cubic box $\Lambda \subset \mathbb{R}^\nu$ of volume $V = L^\nu$, $\nu \geq 3$ with periodic boundary conditions for the wave functions, is described by the full two-body interaction Hamiltonian

$$H_L = \sum_k \epsilon_k a^*_{L,k} a_{L,k} + \frac{1}{2V} \sum_{q,k,k'} v(q) a^*_{L,k+q} a^*_{L,k'-q} a_{L,k'} a_{L,k} - \mu_L N_L, \qquad (1)$$

where the sum runs over the set $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^\nu$ and $\epsilon_k = \frac{|k|^2}{2m}$; $a^\sharp_{L,k}$ are the boson creation/annihilation operators in the one-particle state $\psi_{L,k}(x) = V^{-1/2}e^{ik.x}$, $x \in \Lambda$, $k \in \Lambda^*$, i.e.

$$a_{L,k} = \int_\Lambda a(x)\frac{e^{-ik.x}}{V^{1/2}}dx, \tag{2}$$
$$[a(x), a^*(y)] = \delta(x-y)$$

and $v(q) = \int_{\mathbb{R}^\nu} e^{-iq.x}\phi(x)dx$, ϕ is the periodically extended two-body interaction potential.

The generator of the gauge symmetry is the total number operator N_L, with generator density $a^*(x)a(x)$:

$$N_L = \int_\Lambda a^*(x)a(x)dx.$$

The common choice of order parameter is $V^{-1/2}a^\sharp_{L,0}$, or taking a self-adjoint combination

$$A_L = \frac{i}{\sqrt{2V}}(a^*_{L,0} - a_{L,0}) = \frac{i}{\sqrt{2}V}\int_\Lambda (a^*(x) - a(x))dx,$$

so the order parameter density is given by $\frac{i}{\sqrt{2}}(a^*(x) - a(x))$. One has of course.

$$[N_L, A_L] = \frac{i}{\sqrt{2V}}(a^*_{L,0} + a_{L,0}) \stackrel{V\to\infty}{\longrightarrow} i\sqrt{2\rho_0}\cos\alpha,$$

where ρ_0 is the density of the condensate and α the phase,i.e.

$$V^{-1/2}a^*_{L,0} \to \sqrt{\rho_0}e^{i\alpha}.$$

We are here interested in the behaviour of the q-mode fluctuation ($q \neq 0$) [15] of this generator and order parameter, i.e.

$$F_{L,q}(N) = \frac{1}{V^{1/2}}\int_\Lambda a^*(x)a(x)e^{iq.x}dx \tag{3}$$
$$F_{L,q}(A) = \frac{i}{\sqrt{2}V^{1/2}}\int_\Lambda (a^*(x) - a(x))e^{iq.x}dx, \tag{4}$$

which satisfy the same commutation relation as N_L and A_L:

$$[F_{L,q}(N), F_{L,-q}(A)] = \frac{i}{\sqrt{2V}}(a^*_{L,0} + a_{L,0}) \stackrel{V\to\infty}{\longrightarrow} i\sqrt{2\rho_0}\cos\alpha. \tag{5}$$

In fact we take a sequence $0 \neq q_L \in \Lambda^*$ converging to q so that there is no need to substract expectation values, since for $q \in \Lambda^*$, $\int_\Lambda e^{iq.x}dx = V\delta_{q,0}$, and so that we can also write

$$F_{L,q}(N) = \frac{1}{V^{1/2}} \sum_k a^*_{L,k+q} a_{L,k} \tag{6}$$

$$F_{L,q}(A) = \frac{i}{\sqrt{2}} (a^*_{L,q} - a_{L,-q}). \tag{7}$$

Our first goal will be to define these operators in the thermodynamic limit $L \to \infty$. This will be done via a central limit theorem, as defined in [11, 12]. Afterwards we will be interested in the long wavelength - low frequency limit $q \to 0$ in which collective behaviour is to be expected. In this limit $q \to 0$ we give a connection with the abstractly studied fluctuation operators [16] of the type

$$F_\delta(O) = \lim_{L\to\infty} F_{L,\delta}(O) = \frac{1}{V^{\frac{1}{2}+\delta}} \int_\Lambda (O(x) - < O(x) >) dx, \tag{8}$$

where O is some operator density and δ a critical exponent describing the degree of abnormality of the fluctuations of O, defined by existence of the variance. If $\delta > 0$ there is ODLRO, if $\delta < 0$, the fluctuation is squeezed.

In an interacting Bose gas this $q \to 0$ behaviour will be mainly determined by the spectrum E_q of the Hamiltonian. The density fluctuation $F_{L,q}(N)$ has another very important property, its commutator with the two-body interaction part of the Hamiltonian vanishes:

$$[U_L, F_{L,q}(N)] = 0, \tag{9}$$

where

$$U_L = \frac{1}{2V} \sum_{q,k,k'} v(q) a^*_{L,k+q} a^*_{L,k'-q} a_{L,k'} a_{L,k}.$$

This is easily seen as follows. Commute in U_L, $a^*_{L,k'-q}$ with $a_{L,k}$. This gives

$$U_L = \frac{1}{2V} \sum_{q,k,k'} v(q) a^*_{L,k+q} a_{L,k} a^*_{L,k'-q} a_{L,k'} - \frac{1}{2V} \sum_{q,k} v(q) a^*_{L,k+q} a_{L,k+q}.$$

In the first term, separate the term $q = 0$ from the rest, use translation invariance in the second term, and observe that U_L can then be written as

$$U_L = \frac{1}{2} \sum_{q\neq 0} v(q) F_{L,q}(N) F_{L,-q}(N) + \frac{v(0)}{2V} N_L^2 - \frac{1}{2}\phi(0) N_L. \tag{10}$$

From this expression, (9) is obvious.

In physics, one encounters essentially two types of Bose condensed systems, namely those with a quadratic excitation spectrum ($E_q \propto |q|^2$, $|q|$ small) and those with a superfluid, linear spectrum ($E_q \propto |q|$, $|q|$ small). We will treat in detail an example of each of these cases.

3 The Imperfect Bose Gas

3.1 The model and equilibrium states

To make things more concrete, we consider as a first example the imperfect or mean field Bose gas [17, 18], specified by the local Hamiltonian H_L with periodic boundary conditions [19]:

$$H_L = T_L - \mu_L N_L + \frac{\lambda}{2V} N_L^2 \tag{11}$$

where $\lambda \in \mathbb{R}^+$, Λ the centered cubic box of side length L in $\mathbb{R}^\nu$, $\nu \geq 3$, $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^\nu$. Remark that, apart from a shift in the chemical potential, this Hamiltonian can be obtained from (1) by taking $v(q) = 0$ if $q \neq 0$ in (10).

Talking about the thermodynamic limit, we mean $L \to \infty$ under the constraint that for all L

$$\frac{\omega_L(N_L)}{V} = \rho, \tag{12}$$

where ρ is any positive number standing for the average density of particles, ω_L is the canonical Gibbs state for (11) at some inverse temperature β. It is proved [19] that $\omega_\beta(.) = \lim_L \omega_L(.)$ exists as a space homogeneous state on the algebra of polynomials in the creation and annihilation operators. It is proved that there exists condensation in the zero ($k = 0$) mode state if ρ is large enough and T is small enough.

The phase transition is accompanied by a spontaneous breaking of the gauge symmetry of (11) in the sense that

$$\lim_{L\to\infty} \omega_L \left(\frac{a^*_{L,0}}{V^{1/2}} a(f) \right) = \rho_0 \hat{f}(0) \ , \ \rho_0 > 0, \tag{13}$$

where $a(f) = \int \overline{f(x)} a(x) dx$ for $f \in C_0^\infty(\mathbb{R}^\nu)$, $\hat{f}$ is the Fourier transform of f and ρ_0 is the condensate density. It is proved in [20] that (13) implies, amongst other things, the breaking of the gauge symmetry.

The limit chemical potential is given by

$$\mu = \lambda\rho, \tag{14}$$

and the dynamics coincides with the dynamics of the free Bose gas. The limit Gibbs state has the following form: for all local observables A,

$$\omega_\beta(A) = \frac{1}{2\pi}\int_0^{2\pi} \omega_\beta^\alpha(A)d\alpha, \tag{15}$$

with

$$\omega_\beta^\alpha\left(e^{i(a(f)+a^*(f))}\right) = \exp\left[-\frac{1}{2}(f,Kf) + 2i\rho_0^{1/2}|\hat{f}(0)|\cos\alpha\right] \tag{16}$$

and

$$\left(\widehat{Kf}\right)(k) = \frac{1}{2}\coth(\frac{\beta\epsilon_k}{2})\hat{f}(k). \tag{17}$$

The states $\omega_\beta^\alpha(\alpha \in [0,2\pi])$ are the extremal equilibrium state components of ω_β with the property that

$$\lim_{L\to\infty}\omega_\beta^\alpha\left(\frac{a^*_{L,0}}{V^{1/2}}\right) = \sqrt{\rho_0}e^{i\alpha}, \tag{18}$$

and as operators in the GNS-representation of ω_β^α, one has also

$$\lim_{L\to\infty}\frac{a^*_{L,0}}{V^{1/2}} = \sqrt{\rho_0}e^{i\alpha}. \tag{19}$$

Remark also that the states $\omega_\beta^\alpha(\alpha \in [0,2\pi])$ are quasi-free states, making the computation of expectation values straightforward.

3.2 Collective Goldstone modes

We now turn our attention to the density and order parameter fluctuations. We consider our system to be in one of the extremal equilibrium states ω_β^α, for some $\alpha \in [0,2\pi]$, and without loss of generality, we take $\alpha = 0$, and denote this state again by ω_β.

For notational convenience, if $\rho_0 \neq 0$, denote

$$\rho_{L,q} = \frac{1}{\sqrt{2\rho_0}}F_{L,q}(N), \quad A_{L,q} = F_{L,q}(A).$$

Then we have

$$[\rho_{L,q}, A_{L,-q}] = \frac{i}{2\sqrt{\rho_0 V}}(a^*_{L,0} + a_{L,0}),$$

and by (19):

$$\lim_{L\to\infty}[\rho_{L,q}, A_{L,-q}] = i. \tag{20}$$

More generally

$$\lim_{L\to\infty}[\rho_{L,q}, A_{L,-q'}] = i\delta_{q,q'}. \tag{21}$$

Let us first calculate the variances of $\rho_{L,q}$ and $A_{L,q}$.

Proposition 1 *We have for $q, q' \neq 0$*

(i)

$$\lim_{L\to\infty}\omega_\beta(\rho_{L,q}\rho_{L,-q'})$$
$$= \delta_{q,q'}\left(\frac{1}{2}\coth\frac{\beta\epsilon_q}{2} + \frac{1}{2\rho_0}\int_{\mathbb{R}^\nu}\frac{dk}{(2\pi)^\nu}\frac{1}{e^{\beta\epsilon_{k+q}}-1}\frac{1}{1-e^{-\beta\epsilon_k}}\right),$$

(ii)

$$\lim_{L\to\infty}\omega_\beta(A_{L,q}A_{L,-q'}) = \delta_{q,q'}\frac{1}{2}\coth\frac{\beta\epsilon_q}{2}.$$

Proof. The proof is a straightforward calculation using the quasi-freeness of the state ω_β, e.g.

$$\begin{aligned}
& \omega_\beta(\rho_{L,q}\rho_{L,-q}) \\
= & \frac{1}{2\rho_0 V}\sum_{k,k'}\omega_\beta(a^*_{L,k+q}a_{L,k}a^*_{L,k'-q}a_{L,k'}) \\
= & \frac{1}{2\rho_0 V}\sum_{k}\omega_\beta(a^*_{L,k+q}a_{L,k+q})\omega_\beta(a_{L,k}a^*_{L,k}) \\
= & \frac{1}{2\rho_0 V}\Big(\omega_\beta(a^*_{L,q}a_{L,q})\omega_\beta(a_{L,0}a^*_{L,0}) + \omega_\beta(a^*_{L,0}a_{L,0})\omega_\beta(a_{L,-q}a^*_{L,-q})\Big) \\
& + \frac{1}{2\rho_0 V}\sum_{-q\neq k\neq 0}\omega_\beta(a^*_{L,k+q}a_{L,k+q})\omega_\beta(a_{L,k}a^*_{L,k}).
\end{aligned}$$

In the limit, this becomes

$$\lim_{L\to\infty}\omega_\beta(\rho_{L,q}\rho_{L,-q}) \;=\; \frac{1}{2}\coth\frac{\beta\epsilon_q}{2} + \frac{1}{2\rho_0}\int_{\mathbb{R}^\nu}\frac{dk}{(2\pi)^\nu}\frac{1}{e^{\beta\epsilon_{k+q}}-1}\frac{1}{1-e^{-\beta\epsilon_k}}.$$

The other case is even easier.

□

From this already a few conclusions can be drawn. First of all, consider the integral in the most relevant case $\nu = 3$:

$$\frac{1}{2\rho_0} \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{1}{e^{\beta \epsilon_{k+q}} - 1} \frac{1}{1 - e^{-\beta \epsilon_k}}.$$

Letting $q \to 0$, this integral clearly diverges due to the contribution of the neighbourhood of $k = 0$. Near $k = 0$ we can write it (up to constants) like

$$\int \frac{dk}{(2\pi)^3} \frac{1}{(k+q)^2 k^2}.$$

Taking q e.g. along the z-axis and changing the variable k to $k' = \frac{k}{|q|}$, it can be seen that this integral diverges like $|q|^{-1}$. Since $\coth \frac{\beta \epsilon_q}{2}$ diverges as $|q|^{-2}$ for $q \to 0$, we see that for small q the variance of $\rho_{L,q}$ is completely dominated by the coth-term.

This divergence implies that we should renormalise both $\rho_{L,q}$ and $A_{L,q}$ in order to get a nontrivial limit $q \to 0$ of their variances, i.e.

$$\begin{aligned} \rho_{L,q} &\to \tilde{\rho}_{L,q} = |q| \rho_{L,q}, \\ A_{L,q} &\to \tilde{A}_{L,q} = |q| A_{L,q}. \end{aligned}$$

But this implies that the commutator

$$\lim_{L \to \infty} [\tilde{\rho}_{L,q}, \tilde{A}_{L,-q}] = i|q|^2,$$

vanishes in the limit $q \to 0$.

On the other hand, if one considers the ground state situation (limit $\beta \to \infty$), the $q \to 0$ analysis yields that the variances

$$\lim_{q \to 0} \omega_\infty(\rho_{L,q} \rho_{L,-q})$$

and

$$\lim_{q \to 0} \omega_\infty(A_{L,q} A_{L,-q})$$

are both finite, and the commutation relation between $\rho_{L,q}$ and $A_{L,-q}$ is non-trivial and canonical.

This is not surprising, as one expects true quantum effects on the level of fluctuations only in the ground state. Critical quantum effects are hidden behind the temperature ($T > 0$) fluctuations.

Moreover, it is tempting to identify this renormalisation in q with the exponent δ of (8) via the relation $|q| \propto L^{-1}$. This relation of course being given

by the fact that the first non-zero q-level in finite volume is $|q| = 2\pi L^{-1}$. In that case we obtain for the density fluctuations $F_{L,q}(N)$ that $\delta = 1/3$ in the condensed phase. In the normal phase, the coth-term would be absent and the integral would be convergent also for $q = 0$ because $e^{\beta\epsilon_k}$ would be replaced by $e^{\beta(\epsilon_k-\alpha)}$, with $\alpha < 0$, hence $\delta = 0$. At the critical point, the coth-term would still be absent but the integral would now be divergent like $|q|^{-1}$ as shown before. This would then give $\delta = 1/6$. These three values for δ are exactly the ones calculated in [21].

Since we will be interested in quantum effects on the level of macroscopic fluctuations we will restrict ourself from now on to the ground state (from now on denoted ω). We redefine $\rho_{L,q}$ and $A_{L,q}$ as self-adjoint operators, and we make the (arbitrary) choice of taking the *cos-fluctuation*:

$$\rho_{L,q} = \frac{1}{\sqrt{\rho_0 V}} \int_\Lambda a^*(x)a(x)\cos(q.x)dx \tag{22}$$

$$A_{L,q} = \frac{i}{\sqrt{V}} \int_\Lambda (a^*(x) - a(x))\cos(q.x)dx, \tag{23}$$

or in momentum space

$$\rho_{L,q} = \frac{1}{2\sqrt{\rho_0 V}} \sum_k (a^*_{L,k+q}a_{L,k} + a^*_{L,k-q}a_{L,k}) \tag{24}$$

$$A_{L,q} = \frac{i}{2}[a^*_{L,q} + a^*_{L,-q} - (a_{L,q} + a_{L,-q})], \tag{25}$$

where the normalization is chosen such that

$$\lim_{L\to\infty} [\rho_{L,q}, A_{L,q'}] = i\delta_{q,q'}. \tag{26}$$

It is easy to check that with these definitions

$$\lim_{L\to\infty} \omega(\rho^2_{L,q}) = \frac{1}{2} \tag{27}$$

$$\lim_{L\to\infty} \omega(A^2_{L,q}) = \frac{1}{2}. \tag{28}$$

The rest of this section is devoted to the more mathematical aspects of the realisation of the different fluctuation operators as central limits of operators. The less mathematics minded reader can skip this part at a first reading and proceed immediately to section 3.3.

Let $\mathcal{F}$ be the family of complex continuous functions $f(k, k')$ of two variables $k, k' \in \mathbb{R}^\nu$, satisfying

$$f(\pm k, \pm k') = f(k, k') \tag{29}$$

and

$$\overline{f(k,k')} = f(k',k). \tag{30}$$

With later applications in mind, define for $f, g \in \mathcal{F}$, $\rho_{L,q}(f)$ and $A_{L,q}(g)$ by

$$\begin{aligned}
\rho_{L,q}(f) &= \frac{1}{2\sqrt{\rho_0 V}} \sum_k \Big[f(k+q,k) a^*_{L,k+q} a_{L,k} \qquad (31)\\
&\quad + f(k-q,k) a^*_{L,k-q} a_{L,k} \Big] \\
A_{L,q}(g) &= \frac{i}{2} \Big[g(q,0)(a^*_{L,q} + a^*_{L,-q}) - g(0,q)(a_{L,q} + a_{L,-q}) \Big]. \qquad (32)
\end{aligned}$$

Condition (30) ensures the self-adjointness of these operators. Then define operators $F_{L,q}(f,g)$ by

$$F_{L,q}(f,g) = \rho_{L,q}(f) + A_{L,q}(g). \tag{33}$$

Proposition 2 *For $f, g \in \mathcal{F}$,*

$$\lim_{L\to\infty} \omega\left(F_{L,q}(f,g)^2 \right) = \frac{1}{2} \left| f(q,0) + ig(q,0) \right|^2. \tag{34}$$

Proof. This is a simple calculation using the quasi-freeness of the state ω.

□

In the GNS-representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of the state ω a scalar product is defined by

$$< \pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega >_\omega = \omega(A^* B),$$

and the associated norm is denoted by $\|.\|_\omega$.

Denoting $\pi_\omega(F_{L,q}(f,g))$ again by $F_{L,q}(f,g)$, we then have:

Proposition 3 (A BCH-formula) *Let $f_i, g_i \in \mathcal{F}, i = 1, 2$, then*

$$\lim_{L\to\infty} \left\| e^{iF_{L,q}(f_1,g_1)} e^{iF_{L,q}(f_2,g_2)} - e^{i(F_{L,q}(f_1,g_1)+F_{L,q}(f_2,g_2))} e^{-\frac{1}{2}[F_{L,q}(f_1,g_1),F_{L,q}(f_2,g_2)]} \right\|_\omega$$

$$= 0. \tag{35}$$

Proof. See Appendix A.

□

This result should be compared with the Baker-Campbell-Haussdorff formula which states that for two operators A, B whose commutator is a complex number:

$$e^A e^B = e^{(A+B)} e^{-\frac{1}{2}[A,B]}.$$

This proposition tells us that in a weak sense this BCH-formula remains true for our fluctuation operators, whose commutator becomes a complex number in the thermodynamic limit. Since we are studying fluctuations of *unbounded* operators, the BCH-formula is only true in the GNS-representation of ω. For fluctuations of *bounded* operators, the BCH-formula holds in a much stronger sense, independent of the state (see [15]).

On the complex vectorspace $\mathcal{V}$ of complex linear combinations of elements from $(\mathcal{F}, \mathcal{F})$, define a sesquilinear form $< \cdot | \cdot >_q$ by

$$\begin{aligned} < f_1, g_1 | f_2, g_2 >_q &= \lim_{L\to\infty} \omega \left(F_{L,q}(f_1, g_1)^* F_{L,q}(f_2, g_2) \right) \\ &= \frac{1}{2} \overline{(f_1(q,0) + i g_1(q,0))} \left(f_2(q,0) + i g_2(q,0) \right), \end{aligned} \quad (36)$$

and extension to the whole of $\mathcal{V}$ by linearity.

This form is positive and satisfies the Cauchy-Schwarz inequality by the positivity of ω and the Cauchy-Schwarz inequality for the state ω.

Separating the real and the imaginary part of the restriction of $< \cdot | \cdot >_q$ to the real subspace $(\mathcal{F}, \mathcal{F})$ of $\mathcal{V}$, i.e.

$$< f_1, g_1 | f_2, g_2 >_q = s_q \left(f_1, g_1 | f_2, g_2 \right) + \frac{i}{2} \sigma_q \left(f_1, g_1 | f_2, g_2 \right), \quad (37)$$

defines a real bilinear positive symmetric form s_q and a symplectic form σ_q. (A form σ is called symplectic if $\sigma(x, y) = -\sigma(y, x)$.)

The symplectic form σ_q satisfies

$$\lim_{L\to\infty} [F_{L,q}(f_1, g_1), F_{L,q}(f_2, g_2)] = i \sigma_q \left(f_1, g_1 | f_2, g_2 \right), \quad (38)$$

where the limit is taken in the GNS-representation of ω.

The following proposition is the crucial Central Limit Theorem for the operators $F_{L,q}(f, g)$.

Proposition 4 (Central Limit Theorem) *For $f, g \in \mathcal{F}$, $t \in \mathbb{R}$,*

$$\lim_{L\to\infty} \omega \left(e^{itF_{L,q}(f,g)} \right) = e^{-\frac{t^2}{2} s_q(f,g|f,g)}. \quad (39)$$

Proof. Although a similar theorem could also be proven for temperature states, we will only do it for the ground state, since that is really all we need. In that case, using the quasi-freeness of the state and the fact that all particles are condensed into the zero-energy state simplifies the proof. The details can be found in Appendix B.

□

The C^*-algebra of the canonical commutation relations over (H, σ), with H a real linear space and σ a symplectic form, written as $CCR(H, \sigma)$, is by definition a C^*-algebra generated by elements $\{W(f) : f \in H\}$ such that

(i) $W(-f) = W(f)^*$

(ii) $W(f)W(g) = e^{\frac{i}{2}\sigma(f,g)}W(f+g)$.

Condition (ii) tells us that $W(f)W(0) = W(0)W(f) = W(f)$. Hence $W(0)$ is the unit of the algebra and it follows that $W(f)$ is a unitary for every f. For an elaborate discussion of the CCR, we refer to [22].

Proposition 5 (Reconstruction Theorem) *The linear functional*

$$\tilde{\omega}^q \left(W_q(f,g)\right) = e^{-\frac{1}{2}s_q(f,g|f,g)} \tag{40}$$

defined on the algebra $CCR((\mathcal{F},\mathcal{F}),\sigma_q)$, is a quasi free state.

More explicitly, we have for all $(f_i, g_i) \in (\mathcal{F},\mathcal{F}), i = 1,\ldots,n$,

$$\lim_{L\to\infty} \omega\left(e^{iF_{L,q}(f_1,g_1)}\cdots e^{iF_{L,q}(f_n,g_n)}\right) = \tilde{\omega}^q\left(W_q(f_1,g_1)\cdots W_q(f_n,g_n)\right). \tag{41}$$

The state $\tilde{\omega}^q$ is regular and hence for every (f,g) there exists a self-adjoint Bosonic field $\Phi_q(f,g)$ in the GNS representation $(\mathcal{H}_{\tilde{\omega}^q}, \pi_{\tilde{\omega}^q}, \Omega_{\tilde{\omega}^q})$ such that

$$\pi_{\tilde{\omega}^q}\left(W_q(f,g)\right) = e^{i\Phi_q(f,g)}. \tag{42}$$

This implies that in the sense of the central limit (41), the local fluctuations converge to the Bosonic fields associated with $CCR((\mathcal{F},\mathcal{F}),\sigma_q)$:

$$\text{CLT} - \lim_{L\to\infty} F_{L,q}(f,g) = \Phi_q(f,g). \tag{43}$$

Proof. See Appendix C.

$\square$

The following definitions now clearly make sense:

$$\begin{aligned} \rho_q &= \Phi_q(1,0) = \mathrm{CLT} - \lim_{L\to\infty} \rho_{L,q}, && (44) \\ A_q &= \Phi_q(0,1) = \mathrm{CLT} - \lim_{L\to\infty} A_{L,q}. && (45) \end{aligned}$$

In the same spirit as this central limit, we now define a limit $q \to 0$ of the operators $\Phi_q(f,g)$.

Define a sesquilinear form $< \cdot|\cdot >$ on $\mathcal{V}$ by

$$< f_1, g_1|f_2, g_2 >= \lim_{q\to 0} < f_1, g_1|f_2, g_2 >_q, \tag{46}$$

and a real linear form s and a symplectic form σ in the obvious way:

$$\begin{aligned} s\left(f_1, g_1|f_2, g_2\right) &= \lim_{q\to 0} s_q\left(f_1, g_1|f_2, g_2\right) && (47) \\ \sigma\left(f_1, g_1|f_2, g_2\right) &= \lim_{q\to 0} \sigma_q\left(f_1, g_1|f_2, g_2\right). && (48) \end{aligned}$$

We then get the limit $(q \to 0)$ result:

Proposition 6 (Reconstruction Theorem 2) *The linear functional*

$$\tilde{\omega}\left(W(f,g)\right) = \lim_{q\to 0} \tilde{\omega}^q\left(W_q(f,g)\right) = e^{-\frac{1}{2}s(f,g|f,g)} \tag{49}$$

defined on the algebra $CCR((\mathcal{F},\mathcal{F}),\sigma)$, is a quasi free state.

More explicitly, we have for all $(f_i, g_i) \in (\mathcal{F},\mathcal{F}), i = 1,\ldots,n$,

$$\lim_{q\to 0}\lim_{L\to\infty} \omega\left(e^{iF_{L,q}(f_1,g_1)}\cdots e^{iF_{L,q}(f_n,g_n)}\right) = \tilde{\omega}\left(W(f_1,g_1)\cdots W(f_n,g_n)\right). \tag{50}$$

The state $\tilde{\omega}$ is regular and hence for every (f,g) there exists a self-adjoint Bosonic field $\Phi(f,g)$ in the GNS representation $(\mathcal{H}_{\tilde{\omega}}, \pi_{\tilde{\omega}}, \Omega_{\tilde{\omega}})$ such that

$$\pi_{\tilde{\omega}}\left(W(f,g)\right) = e^{i\Phi(f,g)}. \tag{51}$$

This implies that in this sense of the limit $q \to 0$ (50), the fluctuations Φ_q converge to the Bosonic fields associated with $CCR((\mathcal{F},\mathcal{F}),\sigma)$:

$$\Phi(f,g) = \lim_{q\to 0} \Phi_q(f,g) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} F_{L,q}(f,g). \tag{52}$$

Proof. This is just a matter of taking the limit $q \to 0$ in the different steps of the proof of the previous Proposition.

$\square$

Specifying again to our original operators:

$$\tilde{\rho} = \lim_{q\to 0} \rho_q = \Phi(1,0) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho_{L,q}, \tag{53}$$

$$\tilde{A} = \lim_{q\to 0} A_q = \Phi(0,1) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} A_{L,q}. \tag{54}$$

The algebra of macroscopic fluctuations $CCR((\mathcal{F},\mathcal{F}),\sigma)$ is a coarse grained one, i.e. different microscopic observables can have the same macroscopic fluctuation operators. To describe this mathematically, introduce an equivalence relation $\sim$ on $\mathcal{V}$ by

$$(f_1, g_1) \sim (f_2, g_2) \iff < f_1 - f_2, g_1 - g_2 | f_1 - f_2, g_1 - g_2 >= 0. \tag{55}$$

Another way of stating the equivalence relation is of course

$$(f_1, g_1) \sim (f_2, g_2) \iff \lim_{q\to 0}\lim_{L\to\infty} \omega\left(F_{L,q}(f_1 - f_2, g_1 - g_2)^2\right) = 0. \tag{56}$$

We then have the following result:

Proposition 7 *For $f_i, g_i \in \mathcal{F}, i = 1, 2$, the following are equivalent:*

(i) $(f_1, g_1) \sim (f_2, g_2)$

(ii) $\Phi(f_1, g_1) = \Phi(f_2, g_2)$.

Proof. See Appendix D.

$\square$

A simple example: take $f \in \mathcal{F}$, and define Jf by $(Jf)(q,0) = -if(q,0)$, $(Jf)(0,q) = if(0,q)$ and $(Jf)(k,k') = 0$ for all other values of k en k'. Then $(f,0) \sim (0, Jf)$ or in other words

$$\mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho_{L,q}(f) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} A_{L,q}(Jf). \tag{57}$$

3.3 Dynamics of the collective Goldstone modes

In this section we will derive a dynamics on the level of the macroscopic fluctuations. This dynamics will of course be induced by the microdynamics. Therefore we start with calculating

$$\begin{aligned} i[H_L, \rho_{L,q}] &= \frac{i}{2(\rho_0 V)^{1/2}} \sum_k \left[(\epsilon_{k+q} - \epsilon_k) a^*_{L,k+q} a_{L,k} + (\epsilon_{k-q} - \epsilon_k) a^*_{L,k-q} a_{L,k}\right] \\ &= \rho_{L,q}(\tilde{\epsilon}), \end{aligned} \tag{58}$$

with $\tilde{\epsilon}(k,k') = i(\epsilon_k - \epsilon_{k'})$.

And also

$$\begin{aligned} i[H_L, A_{L,q}] &= -\frac{1}{2}\left(\epsilon_q + (\frac{\lambda}{V} N_L - \mu_L)\right)\left(a^*_{L,q} + a^*_{L,-q} + a_{L,-q} + a_{L,q}\right) \\ &\quad -\frac{i\lambda}{2V}\left(a^*_{L,q} + a^*_{L,-q} - (a_{L,-q} + a_{L,q})\right) \\ &= -A_{L,q}(\tilde{\epsilon}) - \frac{1}{2}\left(\frac{\lambda}{V} N_L - \mu_L\right)\left(a^*_{L,q} + a^*_{L,-q} + a_{L,-q} + a_{L,q}\right) \\ &\quad -\frac{i\lambda}{2V}\left(a^*_{L,q} + a^*_{L,-q} - (a_{L,-q} + a_{L,q})\right) \end{aligned} \tag{59}$$

The second and the third term on the r.h.s. converge to zero as $L \to \infty$, even as operators in the GNS repesentation of ω, so they are of no importance.

Remark that both $\lim_{L\to\infty} \omega(\rho_{L,q}(\tilde{\epsilon})^2) \propto \epsilon_q^2$ and $\lim_{L\to\infty} \omega(A_{L,q}(\tilde{\epsilon})^2) \propto \epsilon_q^2$, so it is natural to define a macroscopic dynamics by

$$\begin{aligned} i[\tilde{H}, \tilde{\rho}] &= \mathrm{CLT} - \lim_{q\to 0} \lim_{L\to\infty} i\left[\frac{1}{\epsilon_q} H_L, \rho_{L,q}\right] \\ &= \mathrm{CLT} - \lim_{q\to 0} \lim_{L\to\infty} \rho_{L,q}\left(\frac{1}{\epsilon_q}\tilde{\epsilon}\right) \\ &= \mathrm{CLT} - \lim_{q\to 0} \lim_{L\to\infty} A_{L,q} \\ &= \tilde{A}, \end{aligned}$$

where we have used equation (57) to go from the second line to the third.

Analogously,

$$\begin{aligned} i[\tilde{H}, \tilde{A}] &= \mathrm{CLT} - \lim_{q\to 0} \lim_{L\to\infty} i\left[\frac{1}{\epsilon_q} H_L, A_{L,q}\right] \\ &= -\mathrm{CLT} - \lim_{q\to 0} \lim_{L\to\infty} A_{L,q}\left(\frac{1}{\epsilon_q}\tilde{\epsilon}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho_{L,q} \\
&= -\tilde{\rho},
\end{aligned}
$$

again using (57).

Hence we have found a canonical pair of observables $\tilde{\rho}$ and $\tilde{A}$, satisfying

$$[\tilde{\rho}, \tilde{A}] = i, \tag{60}$$

which dynamically decouple from the other degrees of freedom of the system, with the dynamics given by

$$
\begin{aligned}
i[\tilde{H}, \tilde{\rho}] &= \tilde{A} \qquad (61)\\
i[\tilde{H}, \tilde{A}] &= -\tilde{\rho}. \qquad (62)
\end{aligned}
$$

So $\tilde{H}$ is the harmonic oscillator Hamiltonian with frequency 1:

$$\tilde{H} = \frac{1}{2}\left(\tilde{\rho}^2 + \tilde{A}^2\right). \tag{63}$$

The virial theorem is also satisfied, i.e.

$$\tilde{\omega}\left(\tilde{\rho}^2\right) = \tilde{\omega}\left(\tilde{A}^2\right). \tag{64}$$

Remark that to go from the microdynamics H_L to the macrodynamics $\tilde{H}$ we had to rescale the Hamiltonian with ϵ_q^{-1}. This should actually be seen as a rescaling of time

$$t \to \tilde{t} = \frac{t}{\epsilon_q},$$

indicating that for small $|q|$ the typical lifetime of a (density) fluctuation with wave length $|q|^{-1}$ is of the order $|q|^{-2}$, becoming infinite in the limit $q \to 0$.

4 The weakly interacting Bose gas

4.1 The model and equilibrium states

Our second model is a model of superfluidity, i.e. with an excitation spectrum E_q linear in $|q|$ for small q. Such a model is provided by [23]. Its Hamiltonian is given by (we take $\nu = 3$ throughout this section)

$$
\begin{aligned}
H_L(c) &= \sum_k \epsilon_k a^*_{L,k} a_{L,k} + \frac{1}{2}\sum_{k\neq 0} v(k)(a^*_{L,k} a^*_{L,-k} c^2 + \bar{c}^2 a_{L,-k} a_{L,k}) \\
&\quad + |c|^2 \sum_{k\neq 0} v(k) a^*_{L,k} a_{L,k} + \frac{v(0)}{2V} N_L^2 - \mu_L N_L, \qquad (65)
\end{aligned}
$$

supplemented with

$$c = \lim_{L\to\infty} \omega_L(V^{-1/2} a_{L,0}), \tag{66}$$

where ω_L is the Gibbs state at some inverse temperature β corresponding to (65).

This Hamiltonian is in fact the original Bogoliubov Hamiltonian for a weakly interacting Bose gas, with an extra term $\frac{v(0)}{2V} N_L^2$ that ensures the superstability of the model.

Again we take the thermodynamic limit under the constraint

$$\lim_{L\to\infty} \frac{1}{V} \omega_L(N_L) = \rho.$$

It is proved in [23] that there exist solutions $\omega_\beta = \lim_{L\to\infty} \omega_L$, for β and ρ large enough, such that

$$\lim_L \omega_\beta(V^{-1/2} a_{L,0}) = c \neq 0,$$

and we will restrict ourself to these solutions.

To describe these equilibrium states we need a Bogoliubov transformation of the operators $a^\sharp_{L,k}$ into new creation and annihilation operators $b^\sharp_{L,k}$:

$$a_{L,k} = b_{L,k} \cosh \alpha_k + b^*_{L,-k} \sinh \alpha_k \tag{67}$$

$$a_{L,-k} = b_{L,-k} \cosh \alpha_k + b^*_{L,k} \sinh \alpha_k, \tag{68}$$

where

$$\tanh 2\alpha_k = -\frac{|c|^2 v(k)}{\epsilon_k + |c|^2 v(k)}. \tag{69}$$

The limit Gibbs state has the same form as in the imperfect Bose gas: for all local observables A,

$$\omega_\beta(A) = \frac{1}{2\pi} \int_0^{2\pi} \omega_\beta^\alpha(A) d\alpha, \tag{70}$$

with

$$\omega_\beta^\alpha \left(e^{i(b(f)+b^*(f))} \right) = \exp \left[-\frac{1}{2}(f, K'f) + 2i|c\hat{f}(0)| \cos \alpha \right] \tag{71}$$

and

$$\left(\widehat{K'f} \right)(k) = \frac{1}{2} \coth \left(\frac{\beta E_k}{2} \right) \hat{f}(k), \tag{72}$$

with $b^\sharp(f)$ the corresponding Bogoliubov transformation of $a^\sharp(f)$ and

$$E_k = \sqrt{(\epsilon_k + |c|^2 v(k))^2 - (|c|^2 v(k))^2} = \sqrt{\epsilon_k(\epsilon_k + 2|c|^2 v(k))}. \tag{73}$$

This is the famous Bogoliubov spectrum which for small k behaves like

$$E_k \simeq \left(\frac{|c|^2 v(0)}{m}\right)^{1/2} |k|.$$

The states $\omega_\beta^\alpha (\alpha \in [0, 2\pi])$ are the quasi-free extremal equilibrium state components of ω_β with the property that

$$\lim_{L\to\infty} \omega_\beta^\alpha \left(\frac{a^*_{L,0}}{V^{1/2}}\right) = |c| e^{i\alpha}, \tag{74}$$

and as operators in the GNS-representation of ω_β^α, one has also

$$\lim_{L\to\infty} \frac{a^*_{L,0}}{V^{1/2}} = |c| e^{i\alpha}. \tag{75}$$

For more details we refer to [23].

As in the imperfect Bose gas, we want to study the density and order parameter fluctuations, $F_{L,q}(N)$ and $F_{L,q}(A)$. However here, the Hamiltonian $H_L(c)$ (65) is a truncation of the full Hamiltonian (1), and due to this truncation, $H_L(c)$ is no longer gauge invariant, i.e.

$$[H_L(c), N_L] \neq 0. \tag{76}$$

The invariance which is left is

$$[H_L(c), N_{L,0}] = 0, \tag{77}$$

with $N_{L,0} = a^*_{L,0} a_{L,0}$ This means that the spontaneously broken symmetry accompanying the phase transition from $c = 0$ to $c \neq 0$ is not the gauge symmetry generated by N_L, but the symmetry generated by $N_{L,0}$.

One example of the implications of this is the following. In the physics literature, the quantity $\lim_{L\to\infty} < F_{L,q}(N) F_{L,-q}(N) >$ is known as the static structure function, usually denoted $S(q)$. It has been known for a long time, both theoretically and experimentally, that at zero temperature this function behaves linearly in q for small q: $S(q) \propto |q|$ (see e.g. [24]). This linear behaviour is essentially due to the fact that $[U_L, F_{L,q}(N)] = 0$ so that

$$< [F_{L,q}(N), [H_L, F_{L,-q}(N)]] > \propto |q|^2.$$

However in our model

$$[H_L(c), N_L] \neq 0,$$

and very much related to this,

$$[U_L(c), F_{L,q}(N)] \neq 0,$$

and indeed it is easy to calculate that here $\lim_{q\to 0} S(q) = \text{const} \neq 0$.

Because of (76) and (77) we expect that this unphysical behaviour is remedied when we replace the total density fluctuations $F_{L,q}(N)$ by condensate density fluctuations $F_{L,q}(N_0)$. However since $N_{L,0}$ can not be written as the integral over some condensate density, it is impossible to define $F_{L,q}(N_0)$ as a usual fluctuation operator. What we want to show now is that it is possible to find a fluctuation operator $F_{L,q}(N_0)$ which behaves mathematically like one expects for a fluctuation operator of the generator of a spontaneously broken symmetry (i.e. we will derive a similar structure as in the imperfect Bose gas) and moreover gives the correct physical behaviour (like e.g.

$$\lim_{L\to\infty} < F_{L,q}(N_0) F_{L,-q}(N_0) > \propto |q|$$

for small q).

In momentum space we have

$$F_{L,q}(N) = \frac{1}{V^{1/2}} \sum_k a^*_{L,k+q} a_{L,k}. \tag{78}$$

This consists of two parts

$$F_{L,q}(N) = \frac{1}{V^{1/2}} \left(a^*_{L,q} a_{L,0} + a^*_{L,0} a_{L,-q} \right) + \frac{1}{V^{1/2}} \sum_{k\neq 0, k+q\neq 0} a^*_{L,k+q} a_{L,k}. \tag{79}$$

The first part

$$\frac{1}{V^{1/2}} \left(a^*_{L,q} a_{L,0} + a^*_{L,0} a_{L,-q} \right)$$

is the part of $F_{L,q}(N)$ which contains the ground state operators $a^\sharp_{L,0}$. It is clearly the fluctuation of the zero-mode particle density, fluctuating to a fixed mode and back to zero. The other part is the fluctuation of the excited modes among each other. Therefore it is natural to define

$$F_{L,q}(N_0) = \frac{1}{V^{1/2}} \left(a^*_{L,q} a_{L,0} + a^*_{L,0} a_{L,-q} \right). \tag{80}$$

The truncation of $F_{L,q}(N)$ to $F_{L,q}(N_0)$ reminds very much the spirit behind the truncation which led to the Bogoliubov approximation of the full Hamiltonian. We can even show how closely those two are related. Take the interaction part U_L of the full Hamiltonian (1) and write it as in (10):

$$U_L = \frac{1}{2} \sum_{k\neq 0} v(k) F_{L,k}(N) F_{L,-k}(N) + \frac{v(0)}{2V} N_L^2 - \frac{1}{2} \phi(0) N_L.$$

Truncate this expression by truncating the operators $F_{L,k}(N)$ to $F_{L,k}(N_0)$ as described above, then:

$$U_L = \frac{1}{2}\sum_{k\neq 0} v(k)F_{L,k}(N_0)F_{L,-k}(N_0) + \frac{v(0)}{2V}N_L^2 - \frac{1}{2}\phi(0)N_L.$$

Write out:

$$\begin{aligned}
&\frac{1}{2}\sum_{k\neq 0} v(k)F_{L,k}(N_0)F_{L,-k}(N_0)\\
&= \frac{1}{2V}\sum_{k\neq 0} v(k)\Big(a_{L,0}a^*_{L,0}a^*_{L,k}a_{L,k} + a^*_{L,0}a_{L,0}a_{L,-k}a^*_{L-k}\\
&\quad + a_{L,0}a_{L,0}a^*_{L,k}a^*_{L,-k} + a^*_{L,0}a^*_{L,0}a_{L,-k}a_{L,k}\Big)\\
&= \frac{1}{2V}\sum_{k\neq 0} v(k)\Big((a_{L,0}a^*_{L,0} + a^*_{L,0}a_{L,0})a^*_{L,k}a_{L,k} + a_{L,0}a_{L,0}a^*_{L,k}a^*_{L,-k}\\
&\quad + a^*_{L,0}a^*_{L,0}a_{L,-k}a_{L,k}\Big) + \frac{\phi(0)}{2}N_{L,0}
\end{aligned}$$

As in [23], replace the operators $\frac{a^\sharp_{L,0}}{V^{1/2}}$ by complex numbers $|c|e^{\pm i\alpha}$ in this part of the interaction and preserve them as operators in the remaining terms, then

$$\begin{aligned}
U_L \;&=\; \frac{1}{2}\sum_{k\neq 0} v(k)|c|^2\left(e^{-2i\alpha}a^*_{L,k}a^*_{L,-k} + e^{2i\alpha}a_{L,-k}a_{L,k}\right)\\
&\quad + |c|^2\sum_{k\neq 0} v(k)a^*_{L,k}a_{L,k} + \frac{v(0)}{2V}N_L^2 - \frac{1}{2}\phi(0)N_L + \frac{\phi(0)}{2}|c|^2V. \qquad (81)
\end{aligned}$$

Apart from the term $\frac{1}{2}\phi(0)N_L$, which only leads to a shift in the chemical potential, and an unimportant constant $\frac{\phi(0)}{2}|c|^2V$ this is exactly the Hamiltonian (65).

4.2 Collective Goldstone modes

We consider again our system to be in one of the extremal equilibrium states ω^α_β, and without loss of generality we take $\alpha = 0$ (i.e. c real), and denote this state by ω_β. As before, let

$$\begin{aligned}
F_{L,q}(N_0) &= \frac{1}{V^{1/2}}\left(a^*_{L,q}a_{L,0} + a^*_{L,0}a_{L,-q}\right)\\
F_{L,q}(A) &= \frac{i}{\sqrt{2}V^{1/2}}\int_\Lambda (a^*(x) - a(x))e^{iq.x}dx\\
&= \frac{i}{\sqrt{2}}(a^*_{L,q} - a_{L,-q}),
\end{aligned}$$

and for ease of notation:

$$\rho^0_{L,q} = \frac{1}{\sqrt{2c^2}} F_{L,q}(N_0), \quad A_{L,q} = F_{L,q}(A).$$

These operators still satisfy the correct commutation relation

$$[\rho^0_{L,q}, A_{L,-q}] = \frac{i}{2\sqrt{c^2 V}} (a^*_{L,0} + a_{L,0}),$$

and by (75):

$$\lim_{L\to\infty} [\rho^0_{L,q}, A_{L,-q}] = i. \tag{82}$$

More generally

$$\lim_{L\to\infty} [\rho^0_{L,q}, A_{L,-q'}] = i\delta_{q,q'}. \tag{83}$$

Proposition 8 *We have for $q, q' \neq 0$*

(i)

$$\lim_{L\to\infty} \omega_\beta(\rho^0_{L,q}\rho^0_{L,-q'}) = \delta_{q,q'} \frac{\epsilon_q}{2E_q} \coth\left(\frac{\beta E_q}{2}\right)$$

(ii)

$$\lim_{L\to\infty} \omega_\beta(A_{L,q}A_{L,-q'}) = \delta_{q,q'} \frac{E_q}{2\epsilon_q} \coth\left(\frac{\beta E_q}{2}\right).$$

Proof. This is an easy calculation using the Bogoliubov transformation (67), (68), the explicit form of the state (71), property (75) and the fact that

$$\begin{aligned} \cosh 2\alpha_q &= \frac{\epsilon_q + c^2 v(q)}{E_q} \\ \sinh 2\alpha_q &= -\frac{c^2 v(q)}{E_q}, \end{aligned}$$

so that

$$\begin{aligned} (\cosh\alpha_q + \sinh\alpha_q)^2 &= \cosh 2\alpha_q + \sinh 2\alpha_q \\ &= \frac{\epsilon_q}{E_q} \\ &= \sqrt{\frac{\epsilon_q}{\epsilon_q + 2c^2 v(q)}} \end{aligned} \tag{84}$$

$$
\begin{aligned}
(\cosh\alpha_q - \sinh\alpha_q)^2 &= \cosh 2\alpha_q - \sinh 2\alpha_q \\
&= \frac{\epsilon_q + 2c^2 v(q)}{E_q} \\
&= \frac{E_q}{\epsilon_q} \\
&= \sqrt{\frac{\epsilon_q + 2c^2 v(q)}{\epsilon_k}}. \qquad (85)
\end{aligned}
$$

This then gives the result via

$$
\begin{aligned}
\lim_{L\to\infty} \omega_\beta(\rho^0_{L,q}\rho^0_{L,-q'}) &= \delta_{q,q'}\frac{1}{2}\lim_{L\to\infty}\omega_\beta\left((a^*_{L,q} + a_{L,-q})(a^*_{L,-q} + a_{L,q})\right) \\
&= \delta_{q,q'}\frac{1}{2}(\cosh\alpha_q + \sinh\alpha_q)^2 \times \\
&\quad \underbrace{\lim_{L\to\infty}\omega_\beta\left((b^*_{L,q} + b_{L,-q})(b^*_{L,-q} + b_{L,q})\right)}_{\coth\left(\frac{\beta E_q}{2}\right)},
\end{aligned}
$$

□

For $\beta < \infty$ the small q-behaviour of these variances is

$$
\begin{aligned}
\lim_{L\to\infty} \omega_\beta(\rho^0_{L,q}\rho^0_{L,-q}) &\simeq \text{const} \neq 0 \\
\lim_{L\to\infty} \omega_\beta(A_{L,q}A_{L,-q}) &\simeq \text{const} \times |q|^{-2}.
\end{aligned}
$$

So again we have the phenomenon that at non-zero temperature it is impossible to do a renormalisation of $\rho^0_{L,q}$ and $A_{L,q}$ which gives both a meaningful $q \to 0$ limit for the variances and preserves a non-trivial commutation relation.

Therefore we will restrict ourself from now on to the ground state (denoted ω). Contrary to the imperfect Bose gas, there remains a non-trivial q-dependence in the ground state. This is because even at zero temperature not all particles condense into the ground state. We have for small q

$$
\begin{aligned}
\lim_{L\to\infty} \omega(\rho^0_{L,q}\rho^0_{L,-q}) &\propto |q| \\
\lim_{L\to\infty} \omega_\beta(A_{L,q}A_{L,-q}) &\propto |q|^{-1}.
\end{aligned}
$$

Remark that for the condensate density fluctuations this is the above mentioned linear behaviour.

We now redefine $\rho^0_{L,q}$ and $A_{L,q}$ to be self-adjoint and renormalised in q, i.e.

$$\rho^0_{L,q} = \frac{1}{2(c^2|q|V)^{1/2}}\left[(a^*_{L,q}+a^*_{L,-q})a_{L,0}+a^*_{L,0}(a_{L,q}+a_{L,-q})\right] \tag{86}$$

$$A_{L,q} = i\frac{|q|^{1/2}}{2}\left[a^*_{L,q}+a^*_{L,-q}-(a_{L,q}+a_{L,-q})\right]. \tag{87}$$

Hence we have still

$$\lim_{L\to\infty}[\rho^0_{L,q}, A_{L,q'}] = i\delta_{q,q'}. \tag{88}$$

It is already interesting at this stage to remark that

$$\lim_{q\to 0}\lim_{L\to\infty}\omega_\beta\left((\rho^0_{L,q})^2\right) = \frac{1}{2\Omega} \tag{89}$$

$$\lim_{q\to 0}\lim_{L\to\infty}\omega_\beta\left(A^2_{L,q}\right) = \frac{\Omega}{2}, \tag{90}$$

with

$$\Omega = \lim_{q\to 0}\frac{E_q|q|}{\epsilon_q} = (4mc^2 v(0))^{1/2}. \tag{91}$$

And hence

$$\Omega^2\lim_{q\to 0}\lim_{L\to\infty}\omega\left((\rho^0_{L,q})^2\right) = \lim_{q\to 0}\lim_{L\to\infty}\omega\left(A^2_{L,q}\right). \tag{92}$$

Again the rest of this section is devoted to the rigorous mathematical treatment of the existence of the fluctuation operators. Again at first reading, the reader can immediately proceed to section 4.3.

With all definitions and notations as above for the imperfect Bose gas, we define again for $f, g \in \mathcal{F}$

$$\begin{aligned}\rho^0_{L,q}(f) &= \frac{1}{2cV^{1/2}}\Big[f(q,0)(a^*_{L,q}+a^*_{L,-q})a_{L,0} \\ &\quad + f(0,q)a^*_{L,0}(a_{L,q}+a_{L,-q})\Big]\end{aligned} \tag{93}$$

$$A_{L,q}(g) = \frac{i}{2}\left[g(q,0)(a^*_{L,q}+a^*_{L,-q})-g(0,q)(a_{L,q}+a_{L,-q})\right], \tag{94}$$

and

$$F_{L,q}(f,g) = \rho^0_{L,q}(f) + A_{L,q}(g). \tag{95}$$

Now we take immediately the double limit $\lim_{q\to 0}\lim_{L\to\infty}$ rather than the two limits separately as for the imperfect Bose gas.

Proposition 9 *For $f, g \in \mathcal{F}$,*

$$\lim_{q\to 0}\lim_{L\to\infty}\omega\left(F_{L,q}(f,g)^2\right)$$

$$= \lim_{q\to 0}\left\{\frac{\epsilon_q + c^2 v(q)}{2E_q}\Big|f(q,0)+ig(q,0)\Big|^2 - \frac{c^2 v(q)}{2E_q}\Re\left[\Big(f(q,0)+ig(q,0)\Big)^2\right]\right\}. \tag{96}$$

Proof. An explicit calculation shows that

$$\lim_{L\to\infty}\omega\left(\rho^0_{L,q}(f)^2\right) = \frac{\epsilon_q + c^2 v(q)}{2E_q}|f(q,0)|^2 - \frac{c^2 v(q)}{4E_q}[f(q,0)^2 + f(0,q)^2],$$

$$\lim_{L\to\infty}\omega\left(A_{L,q}(g)^2\right) = \frac{\epsilon_q + c^2 v(q)}{2E_q}|g(q,0)|^2 + \frac{c^2 v(q)}{4E_q}[g(q,0)^2 + g(0,q)^2]$$

and

$$\begin{aligned}&\omega\left(\rho^0_{L,q}(f)A_{L,q}(g) + A_{L,q}(g)\rho^0_{L,q}(f)\right)\\ =\ & \frac{\epsilon_q + c^2 v(q)}{E_q}\Im[f(q,0)g(0,q)] + \frac{c^2 v(q)}{E_q}\Im[f(q,0)g(q,0)],\end{aligned}$$

which together give (96).

□

From now on we restrict to those pairs of functions (f,g) for which (96) remains finite. This means if f is real, $|f(q,0)|$ should not diverge faster than $|q|^{-1/2}$ and if f is imaginary, $|f(q,0)|$ should go to zero, at least like $|q|^{1/2}$. A real g should satisfy the same condition as an imaginary f and vice versa. Alternatively, we could say we restrict $\mathcal{F}$ to those functions f which satisfy the above conditions and then look at pairs (f, Jg), with f, g in (the restricted) $\mathcal{F}$ and Jg defined as $(Jg)(q,0) = -ig(q,0)$, $(Jg)(0,q) = ig(0,q)$.

Proposition 10 (A BCH-formula) *Let $(f_i, g_i) \in (\mathcal{F}, J\mathcal{F}), i = 1,2$, then*

$$\lim_{L\to\infty}\left\|e^{iF_{L,q}(f_1,g_1)}e^{iF_{L,q}(f_2,g_2)} - e^{i(F_{L,q}(f_1,g_1)+F_{L,q}(f_2,g_2))}e^{-\frac{1}{2}[F_{L,q}(f_1,g_1),F_{L,q}(f_2,g_2)]}\right\|_\omega = 0. \tag{97}$$

Proof. See Appendix A.

$\square$

On $\mathcal{V}$, the space of complex linear combinations of elements from $(\mathcal{F}, J\mathcal{F})$, $J\mathcal{F}$ in the above defined sense, define a positive sesquilinear form $< \cdot|\cdot >$ by

$$< f_1, g_1|f_2, g_2 >= \lim_{q\to 0}\lim_{L\to\infty} \omega\Big(F_{L,q}(f_1, g_1)^* F_{L,q}(f_2, g_2)\Big),$$

and extension to $\mathcal{V}$ by linearity. This can be calculated in the same way as (96) is calculated to give the following expression

$$\lim_{L\to\infty} \omega\Big(F_{L,q}(f_1, g_1) F_{L,q}(f_2, g_2)\Big)$$

$$\begin{aligned} = \quad & \frac{\epsilon_q + c^2 v(q)}{2E_q}\Re\left(\overline{[f_1 + ig_1]}[f_2 + ig_2]\right) - \frac{c^2 v(q)}{2E_q}\Re\Big([f_1 + ig_1][f_2 + ig_2]\Big) \\ & + i\Im\left(\overline{[f_1 + ig_1]}[f_2 + ig_2]\right). \end{aligned}$$

where we adopted the short hand notation $f_i = f_i(q, 0)$, $\bar{f}_i = f_i(0, q)$, etc.

In practice all operators we use, have either $f = 0$ or $g = 0$, and f and g will always be either real or imaginary. It is easily seen that this leads to a considerable simplification of the above formula's.

Separating the real and the imaginary part of the restriction of $< \cdot|\cdot >$ to the real subspace $(\mathcal{F}, J\mathcal{F})$ of $\mathcal{V}$,

$$< f_1, g_1|f_2, g_2 >= s(f_1, g_1|f_2, g_2) + \frac{i}{2}\sigma(f_1, g_1|f_2, g_2), \tag{98}$$

defines a real bilinear positive symmetric form s and a symplectic form σ.

Remark that, as it should, this σ is the same as the one for the imperfect Bose gas since it also satisfies

$$\lim_{q\to 0}\lim_{L\to\infty} [F_{L,q}(f_1, g_1), F_{L,q}(f_2, g_2)] = i\sigma(f_1, g_1|f_2, g_2), \tag{99}$$

where the limit is taken in the GNS-representation of the equilibrium state ω. This commutator is of course independent of the given model since in both models we have

$$V^{-1/2} a^\sharp_{L,0} \to c.$$

Proposition 11 (Central Limit Theorem) *For $(f,g) \in (\mathcal{F}, J\mathcal{F})$, $t \in \mathbb{R}$*

$$\lim_{q\to 0}\lim_{L\to\infty} \omega\left(e^{itF_{L,q}(f,g)}\right) = e^{-\frac{t^2}{2}s(f,g|f,g)}. \tag{100}$$

Proof. Using (75) we get

$$\lim_{q\to 0}\lim_{L\to\infty} \omega\left(e^{itF_{L,q}(f,g)}\right) =$$

$$\lim_{q\to 0}\lim_{L\to\infty} \omega\left(e^{\frac{it}{2}\left\{[f(q,0)+ig(q,0)]\left(a^*_{L,q}+a^*_{L,-q}\right)+\overline{[f(q,0)+ig(q,0)]}\left(a_{L,q}+a_{L,-q}\right)\right\}}\right). \tag{101}$$

With the explicit expression of the state (71), the r.h.s. of (101) can easily be computed and we get (100).

□

Proposition 12 (Reconstruction Theorem) *The linear functional*

$$\tilde{\omega}\left(W(f,g)\right) = e^{-\frac{1}{2}s(f,g|f,g)} \tag{102}$$

defined on the algebra $CCR((\mathcal{F}, J\mathcal{F}), \sigma)$, is a quasi free state.

More explicitly, we have for all $(f_i, g_i) \in (\mathcal{F}, J\mathcal{F}), i = 1, \ldots, n$,

$$\lim_{q\to 0}\lim_{L\to\infty} \omega\left(e^{iF_{L,q}(f_1,g_1)} \cdots e^{iF_{L,q}(f_n,g_n)}\right) = \tilde{\omega}\left(W(f_1,g_1)\cdots W(f_n,g_n)\right). \tag{103}$$

The state $\tilde{\omega}$ is regular and hence for every (f,g) there exists a self-adjoint Bosonic field $\Phi(f,g)$ in the GNS representation $(\mathcal{H}_{\tilde{\omega}}, \pi_{\tilde{\omega}}, \Omega_{\tilde{\omega}})$ such that

$$\pi_{\tilde{\omega}}\left(W(f,g)\right) = e^{i\Phi(f,g)}. \tag{104}$$

This implies that in this sense of the central limit (103), the local fluctuations converge to the Bosonic fields associated with $CCR((\mathcal{F}, J\mathcal{F}), \sigma)$:

$$\Phi(f,g) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} F_{L,q}(f,g). \tag{105}$$

Proof. See the proof of Proposition 5 and Proposition 6.

□

Specifying to our original operators:

$$\begin{aligned} \tilde{\rho}^0 &= \Phi(|q|^{-1/2}, 0) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho^0_{L,q} & (106) \\ \tilde{A} &= \Phi(0, |q|^{1/2}) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} A_{L,q}, & (107) \end{aligned}$$

adopting the notation $(f, g) = (f(q, 0), g(q, 0))$.

The equivalence property also remains true in this case of course, i.e.

$$\begin{aligned} \Phi(f_1, g_1) = \Phi(f_2, g_2) &\iff (f_1, g_1) \sim (f_2, g_2) \\ &\iff \lim_{q\to 0}\lim_{L\to\infty} \omega\left(F_{L,q}(f_1 - f_2, g_1 - g_2)^2\right) = 0, \end{aligned}$$

in particular

$$\mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho^0_{L,q}(f) = \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} A_{L,q}(Jf). \qquad (108)$$

4.3 Dynamics of the collective Goldstone modes

We will now show that also for the weakly interacting Bose gas the fluctuations of the generator and of the order parameter of the SSB decouple dynamically from the other degrees of freedom of the system, and form a harmonic oscillator system.

At the local level, we have

$$\begin{aligned} & i[H_L(c), \rho^0_{L,q}] \\ = \; & \frac{i\epsilon_q}{2(c^2|q|V)^{1/2}} \left[(a^*_{L,q} + a^*_{L,-q})a_{L,0} - a^*_{L,0}(a_{L,q} + a_{L,-q})\right] \\ & + \frac{ic^2 v(q)}{2(c^2|q|V)^{1/2}} \left[(a^*_{L,q} + a^*_{L,-q})(a_{L,0} - a^*_{L,0}) + (a_{L,0} - a^*_{L,0})(a_{L,q} + a_{L,-q})\right] \\ = \; & \rho^0_{L,q}\left(\frac{i\epsilon_q}{|q|^{1/2}}\right) + \frac{ic^2 v(q)}{2(c^2|q|V)^{1/2}} \left[(a^*_{L,q} + a^*_{L,-q})(a_{L,0} - a^*_{L,0})\right. \\ & \left. + (a_{L,0} - a^*_{L,0})(a_{L,q} + a_{L,-q})\right]. \qquad (109) \end{aligned}$$

Since the operators $V^{-1/2}(a_{L,0} - a^*_{L,0})$ converge to 0 in the GNS representation of ω, we only have to take into account the term $\rho^0_{L,q}\left(\frac{i\epsilon_q}{|q|^{1/2}}\right)$. This is what we mentioned earlier, the interaction part of $H_L(c)$ commutes with the condensate density fluctuations as one expects physically, but only in the thermodynamic limit.

Also:

$$\begin{aligned} & -i[H_L(c), A_{L,q}] \\ = & \frac{(\epsilon_q + 2c^2 v(q))|q|^{1/2}}{2} \left[a^*_{L,q} + a^*_{L,-q} + a_{L,q} + a_{L,-q}\right] \\ = & A_{L,q}\left(-i(\epsilon_q + 2c^2 v(q))|q|^{1/2}\right). \end{aligned} \tag{110}$$

As in the imperfect Bose gas, we define a macroscopic dynamics by first renormalizing the Hamiltonian with its spectrum, and then taking the limit. $\mathrm{CLT} - \lim_{q\to 0} \lim_{L\to\infty}$ of the commutators.

$$\begin{aligned} i[\tilde{H}, \tilde{\rho}^0] &= \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} i\left[\frac{1}{E_q} H_L(c), \rho^0_{L,q}\right] \\ &= \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho^0_{L,q}\left(\frac{i\epsilon_q}{E_q|q|^{1/2}}\right) \\ &= \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} A_{L,q}\left(\frac{\epsilon_q}{E_q|q|}|q|^{1/2}\right) \\ &= \frac{1}{\Omega}\tilde{A} \end{aligned} \tag{111}$$

(where we used the equivalence relation). Remember that $A_{L,q} = A_{L,q}(|q|^{1/2})$ and that Ω was defined in (91),

$$\Omega = (4mc^2 v(0))^{1/2}.$$

$$\begin{aligned} -i[\tilde{H}, \tilde{A}] &= \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} -i\left[\frac{1}{E_q} H_L(c), A_{L,q}\right] \\ &= \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} A_{L,q}\left(\frac{-i(\epsilon_q + 2c^2 v(q))|q|^{1/2}}{E_q}\right) \\ &= \mathrm{CLT} - \lim_{q\to 0}\lim_{L\to\infty} \rho^0_{L,q}\left(\frac{E_q|q|}{\epsilon_q}\frac{1}{|q|^{1/2}}\right) \\ &= \Omega\tilde{\rho}^0. \end{aligned} \tag{112}$$

Again remember $\rho^0_{L,q} = \rho^0_{L,q}(|q|^{-1/2})$.

Hence we have found a canonical pair $\tilde{\rho}^0$ and $\tilde{A}$, satisfying

$$[\tilde{\rho}^0, \tilde{A}] = i, \tag{113}$$

which dynamically decouple from the other degrees of freedom of the system, with the dynamics given by

$$i[\tilde{H}, \tilde{\rho}^0] = \frac{1}{\Omega}\tilde{A} \tag{114}$$

$$-i[\tilde{H}, \tilde{A}] = \Omega\tilde{\rho}^0. \tag{115}$$

The solution of these equations is the harmonic oscillator with energy Ω:

$$\tilde{H} = \frac{1}{2\Omega}\left(\Omega^2(\tilde{\rho}^0)^2 + \tilde{A}^2\right). \tag{116}$$

The state $\tilde{\omega}$ is an equilibrium ground state for the Hamiltonian $\tilde{H}$ satisfying the following virial theorem

$$\Omega^2\tilde{\omega}\left((\tilde{\rho}^0)^2\right) = \omega\left(\tilde{A}^2\right). \tag{117}$$

We classified the programme of Anderson [13, 14] about the construction of the canonical Goldstone coordinates.

Remark that also in this case we have done a rescaling of time

$$t \to \tilde{t} = \frac{t}{E_q}$$

when going from the microdynamics $H_L(c)$ to the macrodynamics $\tilde{H}$, indicating again that the typical lifetime of the fluctuations becomes infinite as $q \to 0$, this time with a rate $|q|^{-1}$, as a consequence of the fact that $E_q \propto |q|$ for small q.

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Appendices

A Proof of Proposition 3 and 10

In this proof we make use of the following general formula

$$e^{it(x+y)} = e^{itx} + \int_0^t ds e^{isx} iy e^{i(t-s)(x+y)}, \tag{118}$$

where x, y are self-adjoint operators. From this formula, one deduces useful properties like for example

$$[e^{ix}, z] = i \int_0^1 dt e^{itx} [x, z] e^{i(1-t)x}, \tag{119}$$

$$\|[e^{ix}, z]\| \leq \|[x, z]\|, \tag{120}$$

where x, y are self-adjoint, z arbitrary and $\|.\|$ is some norm.

Denote $F_i = F_{L,q}(f_i, g_i)$, $i = 1, 2$. We need to prove

$$\lim_{L\to\infty} \|e^{iF_1} e^{iF_2} - e^{i(F_1+F_2)} e^{-\frac{1}{2}[F_1,F_2]}\|_\omega = 0.$$

One has

$$\begin{aligned}
& \|e^{iF_1} e^{iF_2} - e^{i(F_1+F_2)} e^{-\frac{1}{2}[F_1,F_2]}\|_\omega \\
= \ & \omega\left(\{e^{iF_1} e^{iF_2} - e^{i(F_1+F_2)} e^{-\frac{1}{2}[F_1,F_2]}\}^* \{e^{iF_1} e^{iF_2} - e^{i(F_1+F_2)} e^{-\frac{1}{2}[F_1,F_2]}\}\right) \\
= \ & 2 - \omega\left(e^{-iF_2} e^{-iF_1} e^{i(F_1+F_2)} e^{-\frac{1}{2}[F_1,F_2]}\right) - \omega\left(e^{\frac{1}{2}[F_1,F_2]} e^{-i(F_1+F_2)} e^{iF_1} e^{iF_2}\right).
\end{aligned}$$

Hence it is sufficient to show

$$\lim_{L\to\infty} \omega\left(e^{-iF_2} e^{-iF_1} e^{i(F_1+F_2)} e^{-\frac{1}{2}[F_1,F_2]}\right) = 1.$$

Define a function $f(t)$ by

$$f(t) = \omega\left(e^{-iF_2} e^{-itF_1} e^{i(tF_1+F_2)} e^{-\frac{t}{2}[F_1,F_2]}\right) - 1.$$

We have to show

$$\lim_{L\to\infty} |f(1)| = 0.$$

By Taylor's theorem, there exists some $t \in [0, 1]$ such that

$$f(1) = f(0) + f'(t) = f'(t),$$

since $f(0) = 0$. Let us calculate $f'(t)$ using (118),

$$\frac{d}{dt} e^{i(tF_1+F_2)} = i \int_0^1 ds e^{is(tF_1+F_2)} F_1 e^{i(1-s)(tF_1+F_2)}.$$

Hence

$$\begin{aligned}
f'(t) \ = \ & \omega\Big(-ie^{-iF_2} e^{-itF_1} F_1 e^{i(tF_1+F_2)} e^{-\frac{t}{2}[F_1,F_2]} \\
& +ie^{-iF_2} e^{-itF_1} \int_0^1 ds e^{is(tF_1+F_2)} F_1 e^{i(1-s)(tF_1+F_2)} e^{-\frac{t}{2}[F_1,F_2]}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}e^{-iF_2}e^{-itF_1}e^{i(tF_1+F_2)}[F_1,F_2]e^{-\frac{t}{2}[F_1,F_2]}\Big) \\
= \;& \omega\Big(-ie^{-iF_2}e^{-itF_1}F_1e^{i(tF_1+F_2)}e^{-\frac{t}{2}[F_1,F_2]} \\
& +ie^{-iF_2}e^{-itF_1}F_1e^{i(tF_1+F_2)}e^{-\frac{t}{2}[F_1,F_2]} \\
& +ie^{-iF_2}e^{-itF_1}\int_0^1 ds[e^{is(tF_1+F_2)},F_1]e^{i(1-s)(tF_1+F_2)}e^{-\frac{t}{2}[F_1,F_2]} \\
& -\frac{1}{2}e^{-iF_2}e^{-itF_1}e^{i(tF_1+F_2)}[F_1,F_2]e^{-\frac{t}{2}[F_1,F_2]}\Big) \\
= \;& \omega\Big(ie^{-iF_2}e^{-itF_1}\int_0^1 ds\Big\{[e^{is(tF_1+F_2)},F_1]e^{i(1-s)(tF_1+F_2)} \\
& -ise^{i(tF_1+F_2)}[F_2,F_1]\Big\}e^{-\frac{t}{2}[F_1,F_2]}\Big).
\end{aligned}
$$

Denote

$$
A=\int_0^1 ds\Big\{[e^{is(tF_1+F_2)},F_1]e^{i(1-s)(tF_1+F_2)}-ise^{i(tF_1+F_2)}[F_2,F_1]\Big\}.
$$

$$
\begin{aligned}
f'(t) &= \omega\Big(ie^{-iF_2}e^{-itF_1}Ae^{-\frac{t}{2}[F_1,F_2]}\Big) \\
&= \omega\Big(ie^{-iF_2}e^{-itF_1}e^{-\frac{t}{2}[F_1,F_2]}A+ie^{-iF_2}e^{-itF_1}[A,e^{-\frac{t}{2}[F_1,F_2]}]\Big),
\end{aligned}
$$

then

$$
|f'(t)|\le\Big|\omega\Big(e^{-iF_2}e^{-itF_1}e^{-\frac{t}{2}[F_1,F_2]}A\Big)\Big|+\Big|\omega\Big(e^{-iF_2}e^{-itF_1}[A,e^{-\frac{t}{2}[F_1,F_2]}]\Big)\Big|. \tag{121}
$$

The first part is estimated as follows:

$$
\Big|\omega\Big(e^{-iF_2}e^{-itF_1}e^{-\frac{t}{2}[F_1,F_2]}A\Big)\Big|\le\|A\|_\omega,
$$

by the Cauchy-Schwarz inequality. We now make an estimation of $\|A\|_\omega$:

$$
\begin{aligned}
\|A\|_\omega &= \Big\|\int_0^1 ds\Big\{[e^{is(tF_1+F_2)},F_1]e^{i(1-s)(tF_1+F_2)}-ise^{i(tF_1+F_2)}[F_2,F_1]\Big\}\Big\|_\omega \\
&\le \int_0^1 ds\Big\|[e^{is(tF_1+F_2)},F_1]e^{i(1-s)(tF_1+F_2)}-ise^{i(tF_1+F_2)}[F_2,F_1]\Big\|_\omega \\
&= \int_0^1 ds\Big\|\int_0^1 dre^{irs(tF_1+F_2)}is[F_2,F_1]e^{i(1-r)s(tF_1+F_2)}e^{i(1-s)(tF_1+F_2)} \\
&\quad -ise^{i(tF_1+F_2)}[F_2,F_1]\Big\|_\omega \\
&= \int_0^1 ds\Big\|\int_0^1 dre^{irs(tF_1+F_2)}is[F_2,F_1]e^{i(1-rs)(tF_1+F_2)} \\
&\quad -ise^{i(tF_1+F_2)}[F_2,F_1]\Big\|_\omega
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 ds \left\| \int_0^1 drise^{irs(tF_1+F_2)} \left[[F_2, F_1], e^{i(1-rs)(tF_1+F_2)} \right] \right\|_\omega \\
&\leq \int_0^1 ds \int_0^1 drs \left\| \left[[F_2, F_1], e^{i(1-rs)(tF_1+F_2)} \right] \right\|_\omega \\
&\leq \int_0^1 ds \int_0^1 drs(1-rs) \left\| [[F_2, F_1], tF_1 + F_2] \right\|_\omega \\
&= \frac{1}{3} \left\| [[F_2, F_1], tF_1 + F_2] \right\|_\omega ,
\end{aligned}
$$

where we have used (119) in the third step and (120) in the fifth. But because the commutator $[F_2, F_1]$ converges to a complex number in the state ω, it can be calculated explicitly that the commutator $[[F_2, F_1], tF_1 + F_2]$ converges to 0. By a similar argument it can also be shown that the second part of (121) converges to 0, thus proving Proposition 3.

□

B Proof of Proposition 4

Take some arbitrary $F_{L,q}(f,g)$ and first of all remark that it can be written as

$$F_{L,q}(f,g) = B_q^* + B_q,$$

where $B_q^* = B_{-q}$ and $[B_q^*, B_q] = 0$ (we have only written the q-dependence explicitly, all other dependencies are implicit). One can for instance take

$$B_q = \frac{1}{2(\rho_0 V)^{1/2}} \sum_k f(k+q,k) a_{L,k+q}^* a_{L,k} + \frac{i}{2} \left(g(q,0) a_{L,q}^* - g(0,q) a_{L,-q} \right).$$

B_q itself can be decomposed into

$$B_q = B_q^0 + \bar{B}_q, \tag{122}$$

with

$$
\begin{aligned}
B_q^0 &= \frac{1}{2(\rho_0 V)^{1/2}} \left(f(q,0) a_{L,q}^* a_{L,0} + f(0,q) a_{L,0}^* a_{L,-q} \right) \\
&\quad + \frac{i}{2} \left(g(q,0) a_{L,q}^* - g(0,q) a_{L,-q} \right) \\
&= \frac{1}{2} \left[\left(f(q,0) \frac{a_{L,0}}{(\rho_0 V)^{1/2}} + ig(q,0) \right) a_{L,q}^* \right. \\
&\quad \left. + \left(f(0,q) \frac{a_{L,0}^*}{(\rho_0 V)^{1/2}} - ig(0,q) \right) a_{L,-q} \right].
\end{aligned}
$$

We want to show now that the part $(B_q^0)^* + B_q^0$ is the only part of $F_{L,q}(f,g)$ which gives a non-zero contribution to the expectation value on the l.h.s. of (39). Expanding the exponential in a power series, we have to calculate expectation values of the type

$$\omega\left((B_q^* + B_q)^m\right).$$

This is obviously zero for m odd, and for $m = 2n$ even, the only non-zero terms are those with a number of starred operators equal to unstarred. Because of the commutation $[B_q^*, B_q]$, these are all equal to

$$\omega\left((B_q^*)^n B_q^n\right).$$

Using the decomposition (122) this becomes

$$\omega\left((B_q^*)^n B_q^n\right) = \omega\left(((B_q^0)^*)^n (B_q^0)^n\right) + \text{other terms.}$$

We now prove that these 'other terms' are all necessarily zero because we are working in the ground state. Using the commutation of the $(B_q^0)^\sharp$ with the $(\bar{B}_q)^\sharp$ these terms are all of the form

$$\omega\left(((B_q^0)^*)^i (B_q^0)^j (\bar{B}_q^*)^{n-i} (\bar{B}_q)^{n-j}\right). \tag{123}$$

We have

$$(\bar{B}_q)^{n-j} = \frac{1}{(4\rho_0 V)^{\frac{n-j}{2}}} \sum_{k_1,\ldots,k_{n-j};-q\neq k_l\neq 0} f(k_1+q,k_1)\ldots f(k_{n-j}+q,k_{n-j}) \\ a^*_{L,k_1+q} a_{L,k_1} \ldots a^*_{L,k_{n-j}+q} a_{L,k_{n-j}}.$$

Using this in (123) one gets sums of expectation values which can be computed by using the quasi-freeness of the state ω, i.e. each expectation is a sum of products of one- and two-point correlations, respectivily $\omega(a^\sharp_{L,k})$ and

$$\omega^T(a^\sharp_{L,k} a^\sharp_{L,k'}) = \omega(a^\sharp_{L,k} a^\sharp_{L,k'}) - \omega(a^\sharp_{L,k})\omega(a^\sharp_{L,k'}),$$

of which only the following are non-zero:

$$\begin{aligned} \omega(a^\sharp_{L,0}) &= \sqrt{\rho_0 V} \\ \omega^T(a_{L,k} a^*_{L,k}) &= \omega(a_{L,k} a^*_{L,k}) = 1, \quad k \neq 0. \end{aligned}$$

We show now that each of the expectations to be computed is zero. Take an arbitrary term. First of all, take some $a_{L,0} a^*_{L,q}$ term from one of the B_q^0's. It can only give a non-zero contribution if it is combined with an $a^*_{L,0} a_{L,q}$ term

from one of the $(B_q^0)^*$'s, so at least we need $i = j$. A term $a^\sharp_{L,\pm q}$ from any of the B_q^0's or $(B_q^0)^*$'s can never be combined with a term coming from the $(\bar{B}_q)^\sharp$'s because this would give rise to an expectation of an odd number of creation and annihilation operators, all with a non-zero index, which is zero. Hence in the quasi-free decomposition, the operators coming from $((B_q^0)^*)^j (B_q^0)^j$ and the operators coming from $(\bar{B}_q^*)^{n-j}(\bar{B}_q)^{n-j}$ completely decouple from each other. But a typical factor arising from $(\bar{B}_q^*)^{n-j}(\bar{B}_q)^{n-j}$ is

$$\omega\left(a^*_{L,k_1+q}a_{L,k_1}\dots a^*_{L,k_{n-j}+q}a_{L,k_{n-j}}a^*_{L,l_1-q}a_{L,l_1}\dots a^*_{L,l_{n-j}+q}a_{L,l_{n-j}}\right),$$

which is zero because it always contains at least one factor $\omega(a^*_{L,k}a_{L,k})$, $k \neq 0$. Hence we proved that indeed

$$\omega\left((B_q^*)^n B_q^n\right) = \omega\left(((B_q^0)^*)^n (B_q^0)^n\right),$$

or in other words

$$\lim_{L\to\infty} \omega\left(e^{itF_{L,q}(f,g)}\right) = \lim_{L\to\infty} \omega\left(e^{itF^0_{L,q}(f,g)}\right), \tag{124}$$

where

$$F^0_{L,q}(f,g) = \rho^0_{L,q}(f) + A_{L,q}(g),$$

with

$$\rho^0_{L,q}(f) = \frac{1}{2}\left[\frac{f(q,0)}{(\rho_0 V)^{1/2}}a_{L,0}\left(a^*_{L,q}+a^*_{L,-q}\right) + \frac{f(0,q)}{(\rho_0 V)^{1/2}}a^*_{L,0}\left(a_{L,q}+a_{L,-q}\right)\right].$$

Hence

$$\begin{aligned} F^0_{L,q}(f,g) &= \frac{1}{2}\left\{\left[f(q,0)\frac{a_{L,0}}{(\rho_0 V)^{1/2}} + ig(q,0)\right]\left(a^*_{L,q}+a^*_{L,-q}\right)\right. \\ &\quad \left. + \left[f(0,q)\frac{a^*_{L,0}}{(\rho_0 V)^{1/2}} - ig(0,q)\right]\left(a_{L,q}+a_{L,-q}\right)\right\}. \end{aligned}$$

Using (19) one gets

$$\lim_{L\to\infty} \omega\left(e^{itF^0_{L,q}(f,g)}\right) =$$

$$\lim_{L\to\infty} \omega\left(e^{\frac{it}{2}\left\{[f(q,0)+ig(q,0)]\left(a^*_{L,q}+a^*_{L,-q}\right)+\overline{[f(q,0)+ig(q,0)]}\left(a_{L,q}+a_{L,-q}\right)\right\}}\right). \tag{125}$$

With the explicit expression of the state (16), the r.h.s. of (125) can easily be computed, and together with (124) one gets (39).

$\square$

C Proof of Proposition 5

We prove (41) by induction on $n \in \mathbb{N}_0$. The basis of the induction is the Central Limit Theorem, Proposition 4.

To prove the induction step, assume that (41) holds for some $n \in \mathbb{N}_0$ and fix $(f_i, g_i) \in (\mathcal{F}, \mathcal{F}), i = 1, \ldots, n+1$.

For convenience write

$$e^{iF_{L,q}(f_1,g_1)} \cdots e^{iF_{L,q}(f_{n-1},g_{n-1})} \equiv W_{L,q},$$

and

$$F_{L,q}(f_i, g_i) \equiv F^i_{L,q}.$$

By the Cauchy-Schwarz inequality and the BCH-formula (35)

$$\begin{aligned} & \lim_{L\to\infty} \left| \omega \left(W_{L,q} \left[e^{iF^n_{L,q}} e^{iF^{n+1}_{L,q}} - e^{i(F^n_{L,q}+F^{n+1}_{L,q})} e^{-\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} \right] \right) \right| \\ \leq & \lim_{L\to\infty} \left\| e^{iF^n_{L,q}} e^{iF^{n+1}_{L,q}} - e^{i(F^n_{L,q}+F^{n+1}_{L,q})} e^{-\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} \right\|_\omega = 0. \end{aligned}$$

Use the Cauchy-Schwarz inequality again to derive that

$$\begin{aligned} & \left| \omega \left(W_{L,q} e^{i(F^n_{L,q}+F^{n+1}_{L,q})} e^{-\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} \right) - \omega \left(W_{L,q} e^{i(F^n_{L,q}+F^{n+1}_{L,q})} \right) e^{-\frac{i}{2}\sigma_q(n|n+1)} \right|^2 \\ & = \left| \omega \left(W_{L,q} e^{i(F^n_{L,q}+F^{n+1}_{L,q})} \left[e^{-\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} - e^{-\frac{i}{2}\sigma_q(n|n+1)} \mathbf{1} \right] \right) \right|^2 \\ & \leq \omega \left(\left[e^{\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} - e^{\frac{i}{2}\sigma_q(n|n+1)} \mathbf{1} \right] \left[e^{-\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} - e^{-\frac{i}{2}\sigma_q(n|n+1)} \mathbf{1} \right] \right) \\ & = 2 - \omega \left(e^{\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} \right) e^{-\frac{i}{2}\sigma_q(n|n+1)} - \omega \left(e^{-\frac{1}{2}[F^n_{L,q}, F^{n+1}_{L,q})]} \right) e^{\frac{i}{2}\sigma_q(n|n+1)}. \end{aligned}$$

This expression converges to zero as $L \to \infty$ because of the convergence of the commutator. Combining the induction hypothesis and the above result, one finds that

$$\begin{aligned} \lim_{L\to\infty} \omega \left(W_{L,q} e^{iF^n_{L,q}} e^{iF^{n+1}_{L,q}} \right) &= \lim_{L\to\infty} \omega \left(W_{L,q} e^{i(F^n_{L,q}+F^{n+1}_{L,q})} \right) e^{-\frac{i}{2}\sigma_q(n|n+1)} \\ &= \tilde{\omega}^q \Big(W_q(1) \cdots [W_q(n) + W_q(n+1)] \Big) \\ & \quad e^{-\frac{i}{2}\sigma_q(n|n+1)} \\ &= \tilde{\omega}^q \left(W_q(1) \cdots W_q(n) W_q(n+1) \right). \end{aligned}$$

The last equality results from the CCR algebraic structure of $CCR((\mathcal{F}, \mathcal{F}), \sigma_q)$.

The only thing left to prove is positivity. Take $f_i, g_i \in \mathcal{F}, i = 1, 2$ and use the definitions of σ_q and s_q along with the Cauchy-Schwarz inequality to derive that

$$\begin{aligned} \frac{1}{4}\left|\sigma_q(f_1, g_1|f_2, g_2)\right|^2 &\leq \left|< f_1, g_1|f_2, g_2 >_q\right|^2 \\ \leq < f_1, g_1|f_1, g_1 >_q < f_2, g_2|f_2, g_2 >_q &\leq s_q(f_1, g_1|f_1, g_1)s_q(f_2, g_2|f_2, g_2). \end{aligned}$$

$\square$

D Proof of Proposition 7

Denote

$$W(f_i, g_i) \equiv W_i.$$

Suppose first that (ii) is satisfied, then

$$\left[\pi_{\tilde{\omega}}\left(W_1\right), \pi_{\tilde{\omega}}\left(W_2\right)\right] = 0$$

and hence

$$\sigma(1|2) = 0.$$

Further

$$\begin{aligned} 1 &= \tilde{\omega}\left(W_1 W_2^*\right) = \tilde{\omega}\left(W_1 W_{-2}\right) \\ &= \tilde{\omega}\left(W_{1-2}\right) = e^{-\frac{1}{2}s(1-2|1-2)}, \end{aligned}$$

where we used the notation $(f_1 - f_2, g_1 - g_2) \to 1 - 2$. By the definition of s this means $< 1 - 2|1 - 2 >= 0$, proving (i).

Conversely, suppose $(f_1, g_1) \sim (f_2, g_2)$ then

$$\frac{1}{4}\left|\sigma(1-2|x)\right|^2 \leq < 1-2|1-2 >< x|x >$$

implies that $\sigma(1-2|x) = 0$ for all x, where x denotes an arbitrary element of $(\mathcal{F}, \mathcal{F})$, i.e. $\pi_{\tilde{\omega}}\left(W_{1-2}\right)$ commutes with all elements of $\pi_{\tilde{\omega}}(CCR((\mathcal{F}, \mathcal{F}), \sigma)) \equiv \mathcal{M}$, or $\pi_{\tilde{\omega}}\left(W_{1-2}\right)$ belongs to the commutant $\mathcal{M}'$ of $\mathcal{M}$. Also

$$\begin{aligned} \left\|\left(\pi_{\tilde{\omega}}\left(W_{1-2}\right) - \mathbf{1}\right)\Omega_{\tilde{\omega}}\right\|^2 &= \tilde{\omega}\left((W_{1-2} - \mathbf{1})^*(W_{1-2} - \mathbf{1})\right) \\ &= 2 - \omega\left((W_{1-2}) - \omega\left((W_{2-1}\right)\right. \\ &= 0. \end{aligned}$$

As $\Omega_{\tilde{\omega}}$ is cyclic for $\mathcal{M}$ it is separating for $\mathcal{M}'$. Hence

$$\pi_{\tilde{\omega}}(W_{1-2}) = \mathbf{1}$$

or

$$\pi_{\tilde{\omega}}(W_1) = \pi_{\tilde{\omega}}(W_2)\,.$$

$\square$

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